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Supermultiplets and relativistic problems: I. The free particle with arbitrary spin in a magnetic field

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Abstract. Equations for relativistic particles for arbitrary spin have been of interest since Dirac original work for spin $\frac{1}{2}$, but they involved either bothersome constraints or start with as many Dirac equations as are required to get the derived spin from its original $\frac{1}{2}$ value. We first show that it is possible to have just one equation involving n α's and β's matrices that give possibilities up to $\frac{1}{2}n$ for the spin. We then decompose the α's and β's into direct products of ordinary spin matrices and a new type of them that we call sign spin. The problem reduces then to one in terms of the generators of a U(4) group entirely similar to the one in the spin–isospin theory of nuclear physics and hence the name of supermultiplets in the title. Using then the techniques of the latter we discuss the problem of a free particle in a magnetic field for $n = 1, 2$ and 3 or equivalently eigenvalues for spins $0, \frac{1}{2}$, 1 and $\frac{3}{2}$, and the energies are given as solutions of elementary algebraic equations.

1. Introduction

The equation of a relativistic particle of spin $\frac{1}{2}$ was proposed long ago by Dirac [1] and it had an enormous success in many applications. The extension of the formalism to arbitrary spin has given rise to a veritable flood of papers in the last 50 years. Dirac himself [2] and Fierz and Pauli [3] made proposals, but which were restricted by bothersome constraints. Bargmann and Wigner [4] started not with one but a system of n Dirac type of equations and obtained a particle of spin $\frac{1}{2}n$ by restricting the wave function to the symmetric solution under permutation. Kemmer [5] managed to obtain a Dirac type of equation but only for spins 0 and 1. In fact Mathews [6], under the strict restrictions with which he worked argued that there could be no relativistic equations with spin higher than 1. Bhabha [7] on the other hand again returned to the possibility of arbitrary spin, though connecting them later with representations of SO(5) group as discussed by Krajcik and Nieto [8]. Weinberg [9] derived the Feynman rules for any spin in which the propagators involve matrices that transform like symmetric traceless tensors of rank $2j$. Nikitin [10] and his collaborators deal elegantly with relativistic particles of arbitrary spin in Coulomb and magnetic monopole fields.

In view of the above references, and possible hundreds more that seem less relevant, one could well ask if there is any reason to deal with the subject of a relativistic particle with arbitrary spin with or without interaction. The authors had two main reasons for getting into this crowded field. The first one was that they decided to follow the Barut approach [11] that they used [12] to get a single relativistic equation for a many-body problem, and particularized it to a single particle thus having only one position $\boldsymbol{r}$ and momentum $\boldsymbol{p}$

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vectors but many $\boldsymbol{\alpha}$'s, β's in their equation. The second is that they noticed that $\boldsymbol{\alpha}$'s and β's could be represented by a direct product of the *ordinary* spin $\boldsymbol{s}$, and a new concept with the same properties as the latter that they called *sign* spin $\boldsymbol{t}$. Thus the problem became very similar to the one in nuclear physics in which we have ordinary spin and isospin, and the main symmetry group goes from U(2) to U(4), where the latter is associated with the supermultiplets indicated in the title.

The formalism developed could be applied to any type of interaction but for simplicity, and to get results that involved no approximation, we restricted ourselves to the problem in a constant magnetic field.

We could start with a Lorentz invariant formulation of the paper as we did in a recent publication where we used a time like unit four vector (u_μ), $\mu = 0, 1, 2, 3$ and applied the analysis to a Dirac oscillator interaction [13]. We proceeded then to discuss in detail the energy spectra of the problem as function of the spin of the particle in the definite frame of reference where $(u_\mu) = (1, 0, 0, 0)$.

For the problems to be discussed in this paper we directly analyse them in the last frame of reference, i.e. $(u_\mu) = (1, 0, 0, 0)$ as the extension to an arbitrary one can be done with the boosts indicated in [13].

We shall start by analysing the well known problem [14] of a relativistic particle of spin $\frac{1}{2}$ in a magnetic field as it will provide us, in a novel way, with a number of results that we will require in connection with the problem of arbitrary spin.

2. Relativistic equation for a spin-$\frac{1}{2}$ particle in a magnetic field

The well known equation [14] for a particle with the characteristics indicated in the title is

$$(\boldsymbol{\alpha} \cdot \boldsymbol{B} + \beta)\psi = E\psi \tag{2.1}$$

where we use units $\hbar = m = c = 1$, m is the mass of the particle,

$$\boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix} \qquad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \tag{2.2}$$

where $\boldsymbol{\sigma}$ the vector of Pauli spin matrices,

$$\boldsymbol{B} = \boldsymbol{p} - (e/2)(\boldsymbol{r} \times \boldsymbol{\mathcal{H}}) \tag{2.3}$$

where $\boldsymbol{p}$ is the momentum of the particle, $\boldsymbol{r}$ its position, e its charge and $\boldsymbol{\mathcal{H}}$ the vector associated with the magnetic field.

The matrices $\boldsymbol{\alpha}$, β are of dimension 4×4 but we could convert them into direct products of 2×2 by introducing the definition

$$\hat{I} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad s_1 \equiv \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad s_2 \equiv \frac{1}{2}\begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \qquad s_3 \equiv \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.4}$$

$$\check{I} \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad t_1 \equiv \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad t_2 \equiv \frac{1}{2}\begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix} \qquad t_3 \equiv \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{2.5}$$

Clearly then we have $\boldsymbol{\alpha}$ and β as the direct products

$$\boldsymbol{\alpha} = 4\boldsymbol{s} \otimes t_1 \qquad \beta = 2\hat{I} \otimes t_3 . \tag{2.6}$$

The matrices s_i, $i = 1, 2, 3$ are those of ordinary spin $\frac{1}{2}$, while t_i, $i = 1, 2, 3, \ldots$, which we distinguish by square instead of round brackets, have the same definition as those of s_i

but play a very different role and we shall call them *sign spin* as we shall later see that they are associated with the sign of the energy. The set of matrices (2.4), (2.5) are identical in form to those appearing in supermultiplet theory as introduced by Wigner [15], but in which the t_i, $i = 1, 2, 3$ were interpreted as the components of the isotopic spin.

Going back to equation (2.1) we can see from (2.4), (2.5) that it can be written as

$$\left[4(\boldsymbol{s} \otimes t_1) \cdot \boldsymbol{B} + 2(\hat{I} \otimes t_3)\right] \psi = E\psi \,. \tag{2.7}$$

As is usual in the case of ordinary spin, we could express the ψ in terms of two components with sign spin $\pm\frac{1}{2}$, which we could designate as

$$\psi = \begin{bmatrix} \psi_+ \\ \psi_- \end{bmatrix} \tag{2.8}$$

and in that case from (2.5), equation (2.7) decomposes into two equations

$$2\boldsymbol{s} \cdot [\boldsymbol{p} - (e/2)(\boldsymbol{r} \times \boldsymbol{\mathcal{H}})]\psi_- = (E-1)\psi_+ \tag{2.9}$$

$$2\boldsymbol{s} \cdot [\boldsymbol{p} - (e/2)(\boldsymbol{r} \times \boldsymbol{\mathcal{H}})]\psi_+ = (E+1)\psi_- \,. \tag{2.10}$$

Multiplying the first by $(E+1)$ and substituting in the second, we get for ψ_+ the equation

$$[(p_1^2 + p_2^2 + p_3^3) + (e^2\mathcal{H}^2/4)(x_1^2 + x_2^2) + e\mathcal{H}(x_1p_2 - x_2p_1) + 2e\mathcal{H}s_3]\psi_+ = (E^2-1)\psi_+ \tag{2.11}$$

where we assumed $\mathcal{H}$ to be a constant and in the direction of x_3.

The eigenvalues and eigenfunctions are well known in terms of cylindrical coordinates [14] but we will rewrite it in terms of creation and annihilation operators, to have them in a form more convenient for our later discussion of the problem with arbitrary spin.

Let us start by defining the creation and annihilation operators in the plane (x_1, x_2) as

$$\eta_i = \frac{1}{\sqrt{2}}\left[\left(\frac{e\mathcal{H}}{2}\right)^{1/2} x_i - \mathrm{i}\left(\frac{e\mathcal{H}}{2}\right)^{-1/2} p_i\right] \qquad \xi_i = \frac{1}{\sqrt{2}}\left[\left(\frac{e\mathcal{H}}{2}\right)^{1/2} x_i + \mathrm{i}\left(\frac{e\mathcal{H}}{2}\right)^{-1/2} p_i\right] \tag{2.12}$$

with $i = 1, 2$. Furthermore we introduce spherical components of these operators in the form

$$\eta_\pm = \frac{1}{\sqrt{2}}(\eta_1 \pm \mathrm{i}\eta_2) \qquad \xi^\pm = \frac{1}{\sqrt{2}}(\xi_1 \mp \mathrm{i}\xi_2)\,. \tag{2.13}$$

Equation (2.11) takes then the form

$$\left[e\mathcal{H}(2\eta_+\xi^+ + 1) + p_3^2 + 2e\mathcal{H}s_3\right]\psi_+ = (E^2-1)\psi_+ \,. \tag{2.14}$$

The eigenfunctions of (2.14) can be written in the form of a ket

$$|n_+n_-k\sigma\rangle = \frac{\eta_+^{n_+}\eta_-^{n_-}}{\sqrt{n_+!n_-!}}|0\rangle \mathrm{e}^{\mathrm{i}kx_3}\chi_\sigma \tag{2.15}$$

where

$$|0\rangle = \pi^{-1/2}\exp[-\tfrac{1}{2}(x_1^2 + x_2^2)] \tag{2.16}$$

is the ground state and $n_\pm$ take the integer values $n_\pm = 0, 1, 2, \ldots$. The χ_σ stands for the ordinary spin-$\frac{1}{2}$ function with the projection $\sigma = \pm\frac{1}{2}$.

The eigenvalue of the energies are then

$$E^2_{n_+k\sigma} = 1 + e\mathcal{H}(2n_+ + 1 + 2\sigma) + k^2 \tag{2.17}$$

and it does not depend on n_-, so there is an infinite degeneracy in this quantum number.

As a last point concerning this elementary problem it is useful to express $\boldsymbol{s}\cdot\boldsymbol{B}$ appearing in (2.7) in terms of creation and annihilation operators, and particularly to notice that only those with the + sign, i.e. η_+, ξ^+ will be present. For this purpose we note that in spherical components.

$$\boldsymbol{s}\cdot\boldsymbol{B} = \sum_q (-1)^q s_{-q} B_q \qquad q = +, 0, - \tag{2.18}$$

with

$$s_\pm = \mp\frac{1}{\sqrt{2}}(s_1 \pm \mathrm{i}s_2) \qquad B_\pm = \mp\frac{1}{\sqrt{2}}(B_1 \pm \mathrm{i}B_2) \qquad B_0 = B_3 \qquad s_0 = s_3\,. \tag{2.19}$$

From the definition (2.3) of $\boldsymbol{B}$ and replacing $p_i, x_i, i = 1, 2$ by $\eta_\pm, \xi^\pm$, as follows from (2.12), (2.13), we arrive at the expression

$$\boldsymbol{s}\cdot\boldsymbol{B} = \mathrm{i}(e\mathcal{H})^{1/2}(\eta_+ s_- + \xi^+ s_+) + s_0 p_3\,. \tag{2.20}$$

Matrix elements of this operator with respect to states of the form (2.14), will be relevant for the determination of the energy spectrum as function of the spin of a relativistic particle in a magnetic field.

3. Relativistic equation for a particle of *arbitrary* spin in a magnetic field

As is well known the Dirac equation for a system of n non-interacting particles can be written as

$$\sum_{u=1}^{n}(\boldsymbol{\alpha}_u\cdot\boldsymbol{p}_u + \beta_u)\psi = E\psi \tag{3.1}$$

where

$$\beta_u = I\otimes I\otimes\cdots\otimes I\otimes\beta\otimes I\cdots\otimes I\otimes I \tag{3.2}$$

is a direct product in 4×4 matrices where $n-1$ of them are unity and in the u position we have a β of the form (2.2). A similar definition holds for the $\boldsymbol{\alpha}_u$.

The validity of (3.1) is justified [16] by the fact that by squaring, rearranging, squaring again, etc, and using the anticommuting properties of the $\boldsymbol{\alpha}_u, \beta_u$ we can obtain a 2^n degree algebraic equation involving only the E and the $\boldsymbol{p}_u$ and its 2^n roots turn out to be

$$E = \pm\sqrt{p_1^2+1} \pm \sqrt{p_2^2+1}\cdots \pm\sqrt{p_n^2+1} \tag{3.3}$$

as the Einstein relation leads us to expect.

Nothing prevents us considering the case when all the momenta are equal, i.e. $\boldsymbol{p}_u = \boldsymbol{p}, u = 1, 2, \ldots, n$ and in that case we have an equation for a single particle, but as each $\boldsymbol{\alpha}_u$ is associated with spin $\frac{1}{2}$, the presence of n of them indicate that our particle would, in general, have a mixture of spins with values

$$\tfrac{1}{2}n,\ \tfrac{1}{2}n-1, \ldots, \tfrac{1}{2} \qquad \text{or} \qquad 0\,. \tag{3.4}$$

If such a type of particle is in a magnetic field we just have to replace $\boldsymbol{p}$ by $\boldsymbol{B}$ of (2.3) and have the equation

$$\sum_{u=1}^{n}(\boldsymbol{\alpha}_u\cdot\boldsymbol{B} + \beta_u)\psi = nE\psi\,. \tag{3.5}$$

Note that in (3.5) the energy E of (3.1) is replaced by nE as each of the terms in the summation on the left-hand side of (3.5) makes a contribution and we would like to denote the energy as the average of these contributions. This point was discussed in detail in [13].

We would like the solutions of the equation (3.5) to be characterized by a definite spin, at least for that part of them that have the highest positive value for the energy. This is feasible as Bargmann and Wigner [4] already indicated that in the case of free particles the solution corresponding to the symmetric representation of the permutation group has spin $\frac{1}{2}n$.

Note that equation (3.5) is invariant under permutation of the u indices of the $\boldsymbol{\alpha}_u, \beta_u$ matrices and thus we could characterize our solutions by irreps of the permutation group $S(n)$. This we shall do but with the help of its complementary U(4) group [17, 18].

To begin with we return to the development (2.6) of the $\boldsymbol{\alpha}$ and β but now with an index u, i.e.

$$\boldsymbol{\alpha}_u = 4\boldsymbol{s}_u \otimes t_{1u} \tag{3.6}$$

$$\beta_u = 2\hat{I} \otimes t_{3u} \tag{3.7}$$

where now $\boldsymbol{s}_u, t_{iu}, t_{3u}$ are direct products of n 2×2 matrices, where $n-1$ of them are unity and in the position u appears an s_i or t_i, $i = 1, 2, 3$ of the form (2.4) and (2.5), respectively.

From eqautions (3.6), (3.7) we see that equation (3.5) can be written as

$$\sum_{u=1}^{n} \left\{ 4 \sum_{i=1}^{3} [(s_{iu} \otimes t_{iu}) B_i] + 2\hat{I} \otimes t_{3u} \right\} \psi = nE\psi \tag{3.8}$$

where instead of a scalar product as in (2.7) we prefer to write it in Cartesian components $i = 1, 2, 3$ and sum over them.

We now proceed to show that besides the operator B_i, the other terms appearing in (3.8) are part of the generators of a U(4) group complementary to the permutation symmetry group of the problem. We start by defining

$$S_i \equiv \sum_{u=1}^{n} s_{iu} \otimes \check{I}, \;\; T_i \equiv \sum_{u=1}^{n} \hat{I} \otimes t_{iu}, \;\; R_{ij} \equiv \sum_{u=1}^{n} s_{iu} \otimes t_{ju} \,. \tag{3.9}$$

As we have the commutation relations

$$[s_{iu}, s_{ju}] = \mathrm{i}\epsilon_{ijk} s_{ku} \qquad [s_{iu}, s_{jv}] = 0 \qquad \text{if} \quad u \neq v \tag{3.10}$$

and similarly for the components t_{iu} of the sign spin, while of course $[s_{iu}, t_{jv}] = 0$, we show in the appendix that the operators (3.9) satisfy the commutation relations

$$\begin{aligned} &[S_i, S_j] = \mathrm{i}\epsilon_{ijk} S_k \qquad [T_i, T_j] = \mathrm{i}\epsilon_{ijk} T_k \qquad [S_i, T_j] = 0 \\ &[S_i, R_{jk}] = \mathrm{i}\epsilon_{ij\ell} R_{\ell k} \qquad [T_i, R_{jk}] = \mathrm{i}\epsilon_{ik\ell} R_{j\ell} \\ &[R_{ij}, R_{k\ell}] = \tfrac{1}{4}\mathrm{i}\epsilon_{ikm} S_m \delta_{j\ell} + \tfrac{1}{4}\mathrm{i}\delta_{ik}\epsilon_{j\ell n} T_n \,. \end{aligned} \tag{3.11}$$

From the commutation rules (3.11) we conclude that the operators (3.9) together with the unit operator $\hat{I} \otimes \check{I}$, are the 16 generators of a U(4) group [19], that is complementary to the $S(n)$ group of permutations acting on each ordinary and sign spin that have the index $u = 1, 2, \ldots, n$ [17, 18].

The irreps of U(4) are characterized by a partition $\{h_1 h_2 h_3 h_4\} \equiv \{h\}$ which at the same time is the partition characterizing $S(n)$ and thus [18]

$$h_1 + h_2 + h_3 + h_4 = n \qquad h_1 \geqslant h_2 \geqslant h_3 \geqslant h_4 \geqslant 0 \,. \tag{3.12}$$

To define the states associates with this partition we can use the Gelfand–Zetlin scheme [20], or more conveniently use a chain of subgroups of U(4) that are more relevant to the problem. Clearly one that has this type of property is

$$U(4) \supset \hat{U}(2) \otimes \check{U}(2) \tag{3.13}$$

where Û(2), Ǔ(2) are associated respectively with the ordinary and a sign spin whose generators are $S_i, T_i, i = 1, 2, 3$.

The states are then characterized by the partition $\{h\}$ as well by the eigenvalues of the Casimir operator of the ordinary and sign spin, i.e. $s(s+1), t(t+1)$. Furthermore the U(2) group has an O(2) subgroup whose irreps could be characterized by an index σ in the case of the ordinary spin and τ for the sign spin. Thus the basis of the irreducible representations (BIR) of the chain of group (3.12) can be denoted by the ket

$$|\{h\}\gamma s\sigma t\tau\rangle\ . \tag{3.14}$$

γ is an index that distinguishes the representations (s, t) in $\{h\}$ when they are repeated [18].

The ket (3.13) can be determined in terms of elementary permissible diagrams (EPD) by a procedure similar to that used in [21] for the chain

$$U(3) \supset O(3) \supset O(2) \tag{3.15}$$

and we plan to follow this program in a future publication, but here we only care that this type of ket exists, and in the examples that we shall discuss at the end of this paper, we shall obtain it explicitly in a more elementary manner.

We now return to our basic equation (3.8) which, in the notation (3.9), can be written as

$$\left[4\sum_{i=1}^{3}(R_{i1}B_i) + 2T_3\right]\psi = nE\psi. \tag{3.16}$$

Immediately we note that (3.16) besides B_i that depends on the coordinates, momenta and $\mathcal{H}$, it has only 4 generators of our U(4) group $R_{i1}, T_3, i = 1, 2, 3$ and thus ψ could be characterized by an irrep $\{h\}$ of U(4).To find other integrals of motion we note that (2.20) would apply also if the single spin $\boldsymbol{s}$ is replaced by the total one $\boldsymbol{S} = \sum_{u=1}^{n} \boldsymbol{s}_u$ and thus we see that $\boldsymbol{S}\cdot\boldsymbol{B}$ contains only η_+, ξ^+. Thus as there are no terms η_-, ξ^- the number operator $\boldsymbol{\eta}_-\cdot\boldsymbol{\xi}^-$, is also an integral of motion characterizing the wave function with the eigenvalue n_-. Furthermore as we have chosen the direction of the vector $\boldsymbol{\mathcal{H}}$ as x_3, the total angular momentum in that direction is also an integral of motion, i.e.

$$J_3 = (x_1p_2 - x_2p_1) + \sum_{u=1}^{n} s_{3u} \tag{3.17}$$

and its eigenvalue, that we denote by μ, would also characterize ψ.

Now, as in [13], we need to obtain the set of states in terms of which we can represent the operator on the left-hand side of (3.16) as a finite matrix. To do this we first note that for the configuration part of these states we can use (2.15) suppressing χ_σ, while for the spin part we can employ (3.14) to obtain kets of the form

$$\left(\frac{\eta_+^{n_+}\eta_-^{n_-}}{\sqrt{n_+!n_-!}}|0\rangle\right)\mathrm{e}^{\mathrm{i}kx_3}|\{h\}\gamma s\sigma t\tau\rangle \tag{3.18}$$

which now we wish to rewrite so that the integrals of motion for the operator in the left-hand side of (3.16) appear explicitly.

For this purpose we note that the orbital angular momentum can also be written as

$$L_3 = x_1p_2 - x_2p_1 = \eta_+\xi^+ - \eta_-\xi^- \tag{3.19}$$

so that μ associated with J_3 takes the value

$$\mu = n_+ - n_- + \sigma \ . \tag{3.20}$$

As n_- is also an integral of motion we shall denote from now on as ν and thus $n_+ = \mu+\nu-\sigma$.

Substituting these results in (3.17) we obtain the ket

$$|\mu + \nu - \sigma, \nu, k\{h\}\gamma s\sigma t\tau\rangle = \left(\frac{\eta_+^{\mu+\nu-\sigma}\eta_-^{\nu}}{\sqrt{(\mu+\nu-\sigma)!\nu!}}|0\rangle\right)\mathrm{e}^{\mathrm{i}kx_3}|\{h\}\gamma\sigma t\tau\rangle \ . \tag{3.21}$$

From the above discussion we see that the matrix elements of the operator on the left-hand side of (3.16) with respect to the states (3.21) will be diagonal in the indices $\mu\nu k\{h\}$, and as for a fixed $\{h\}$ we have only a finite number of values for γ, s, t, our matrix will be finite. By diagonalizing it we will get the values of the energy as function of $\mu, \nu, k\{h\}$ and the last one, which is the irrep of the U(4) group, will also give to us information of the spins associated with given energies.

We now need to express the matrix elements of the operators on the left-hand of (3.16) explicitly with respect to the states (3.21). For this purpose its convenient to express the operator in question in spherical rather than Cartesian indices. For this objective we note the relation

$$t_{1u} = \frac{1}{\sqrt{2}}(-t_{+u} + t_{-u}) \tag{3.22}$$

where the index 1 is Cartesian and $\pm$ are spherical. Furthermore, we can write

$$\begin{aligned} H \equiv \sum_{i=1}^{3} 4(R_{i1}B_i) + 2T_3 &= \sum_{u=1}^{n}\left\{4\sum_{i=1}^{3}[(s_{iu}B_i)\otimes t_{iu}]\right\} + 2T_3 \\ &= \sum_{u=1}^{n}\left\{4\left[\mathrm{i}(e\mathcal{H})^{1/2}(\eta_+ s_{-u} + \xi^+ s_{+u}) + s_{ou}p_3\right]\otimes\frac{1}{\sqrt{2}}(-t_{+u}+t_{-u})\right\} + 2T_3 \\ &= 4\mathrm{i}\left(\frac{e\mathcal{H}}{2}\right)^{1/2}\left[\eta_+(-R_{-+}+R_{--}) + \xi^+(-R_{++}+R_{+-})\right] \\ &\quad + \frac{4}{\sqrt{2}}(-R_{0+}+R_{0-})p_3 + 2T_3 \end{aligned} \tag{3.23}$$

where we made use of the fact that the operators B_i, η_+, ξ^+ that depend only on coordinates and momenta of course commute with the sign and ordinary spins t_{iu} and s_{iu}, and we used also the development (2.20) of $\boldsymbol{S}\cdot\boldsymbol{B}$, as well as the notation $+, 0, -$ of spherical components.

The matrix element of H of (3.22) with respect to the states (3.21) can now be obtained straightforwardly by noting that

$$\eta_+|n_+\rangle = \sqrt{n_+ + 1}|n+1\rangle \qquad \xi^+|n_+\rangle = \sqrt{n_+}|n_+ - 1\rangle \qquad p_3|k\rangle = k|k\rangle \tag{3.24}$$

while the more relevant part is to obtain matrix elements of the type

$$\langle\{h\}\gamma' s'\sigma' t'\tau'|R_{qq'}|\{h\}\gamma s\sigma t\tau\rangle = \langle s\sigma, 1q|s'\sigma'\rangle\langle t\tau, 1q'|t\tau'\rangle\langle\{h\}\gamma' s' t'\|R\|\{h\}\gamma st\rangle \tag{3.25}$$

where we made use of the Wigner–Eckart theorem to derive the right-hand side in which $\langle\cdot|\cdot\rangle$ are Clebsch–Gordan coefficients. The reduced matrix element of R can be obtained using fractional parentage and Racah coefficients [22] as it is identical in form to the one appearing in ordinary supermultiplet theory [15], where, besides the ordinary spin, one is dealing with isospin instead of sign spin.

It is also important to obtain the full matrix element of the operator H of (3.23) in the basis (3.21) and for this purpose it is convenient to write H in the form

$$H = \sum_{q,q'} \lambda_{qq'} R_{qq'} + 2T_3 \tag{3.26}$$

where $\lambda_{qq'}$ is defined by

$$\begin{aligned} \lambda_{-+} &= -4\mathrm{i}\omega\eta_+ \qquad \lambda_{--} = 4\mathrm{i}\omega\eta_+ \qquad \lambda_{0+} = -2\sqrt{2}k \\ \lambda_{++} &= -4\mathrm{i}\omega\xi^+ \qquad \lambda_{+-} = 4\mathrm{i}\omega\xi^+ \qquad \lambda_{0-} = 2\sqrt{2}k \end{aligned} \tag{3.27}$$

and where we replaced p_3, which is an integral of motion of the problem, by k in view of the appearance of $\exp(\mathrm{i}kx_3)$ in (3.21). Furthermore, we shall from now on use the notation

$$\omega \equiv (e\mathcal{H}/2)^{1/2} . \tag{3.28}$$

To obtain the secular equation that will give use the different energies associated with the integrals of motion $\mu, \nu, k, \{h\}$ we need to consider from (3.16), (3.23) the numerical matrix

$$\| \langle \mu + \nu - \sigma', \nu k\{h\}\gamma' s'\sigma' t'\tau' | H - nEI | \mu + \nu - \sigma, \nu k\{h\}\gamma s\sigma t\tau \rangle \| . \tag{3.29}$$

We are now, in principle, in a position to determine the spectra $E(\mu\nu k\{h\})$ for any n, which implies also the appearance of states in which the spins can take values up to $\frac{1}{2}n$. For the case $n = 1$ we already obtained the result in (2.17), though we used a simpler procedure than the one outlined in this section. We shall though proceed to illustrate the general supermultiplet method that leads to a secular equation for the energy in the cases of $n = 2$ and 3, where for the first we shall consider both partitions $\{11\}$ and $\{2\}$, while for $n = 3$ we shall only state the result for $\{h\} = \{3\}$ with a brief outline of its derivation.

4. Examples

From equations (3.29), (3.25) we note that the only matrix element in the secular equation for which we do not have an explicit expression is

$$\langle \{h\}\gamma' s' t' \| R \| \{h\}\gamma s t \rangle \tag{4.1}$$

and so we shall proceed to derive it and later substitute it in (3.29).

(a) The n = 2 case

If we have two ordinary and sign spins of values $\frac{1}{2}$ their ket will be

$$\left| \tfrac{1}{2}\tfrac{1}{2}s\sigma, \ \tfrac{1}{2}\tfrac{1}{2}t\tau \right\rangle . \tag{4.2}$$

Clearly for $s = 1$ the ordinary spin part will be symmetric and for $s = 0$ it will be antisymmetric, and the same holds for the value 1 and 0 of t. If we want the ket to be completely symmetric under exchange of both ordinary and sign spin, clearly we either have $s = 1, t = 1$ or $s = 0, t = 0$, while if we want it to be antisymmetric under the same circumstances $s = 1, t = 0$ or $s = 0, t = 1$.

Thus the states corresponding to the partitions $\{2\}, \{11\}$ of S(2), and thus also of U(4) [18], from which we delete the projection of the spins, can be written as

$$|\{2\}st\rangle = |\{2\}11\rangle, |\{2\}00\rangle \tag{4.3}$$

$$|\{11\}st\rangle = |\{11\}10\rangle, |\{11\}01\rangle \tag{4.4}$$

as from complementarity considerations [17] the irreps of S(n) coincide with those of U(4).

Returning to the reduced matrix element (4.1), we note that

$$R_{qq'} = s_q(1) \otimes t_{q'}(1) + s_q(2) \otimes t_{q'}(2) \tag{4.5}$$

and in both the symmetric case {2} and the antisymmetric one {11} we can reduce (4.1) to a factor of 2 by the matrix element of $s_q(1) \otimes t_{q'}(1)$. Note that the index u in s_{iu}, t_{ju}, has now been put in parenthesis as $s_i(u), t_j(u)$, where in this case $u = 1$ or 2. To be able to calculate (4.1) for all four cases (4.3), (4.4) at a single step, we suppress the partition and remember the definition (4.2) of our state so we can write (4.1) as

$$\begin{aligned} 2\langle \tfrac{1}{2}\tfrac{1}{2}s',\ \tfrac{1}{2}\tfrac{1}{2}t' \| s(1) \otimes t(1) \| \tfrac{1}{2}\tfrac{1}{2}s,\ \tfrac{1}{2}\tfrac{1}{2}t\rangle \\ &= 2\langle \tfrac{1}{2}\tfrac{1}{2}s' \| s(1) \| \tfrac{1}{2}\tfrac{1}{2}s\rangle \langle \tfrac{1}{2}\tfrac{1}{2}t' \| t(1) \| \tfrac{1}{2}\tfrac{1}{2}t\rangle \\ &= 3(-1)^{s'+t'}[(2s+1)(2t+1)]^{1/2} W(\tfrac{1}{2}s\tfrac{1}{2}s';\ \tfrac{1}{2}1) W(\tfrac{1}{2}t\tfrac{1}{2}t';\ \tfrac{1}{2}1) \end{aligned} \tag{4.6}$$

where we made use of [23, relation (6.25)], and W is a Racah coefficient.

From the table of Racah coefficients in [23, page 227] we see that the reduced matrix elements of R in (4.6) take the values

$$\begin{aligned} \langle\{11\}10\|R\|\{11\}10\rangle &= 0 \\ \langle\{11\}10\|R\|\{11\}01\rangle &= -\tfrac{1}{2}\sqrt{3} \\ \langle\{11\}01\|R\|\{11\}10\rangle &= -\tfrac{1}{2}\sqrt{3} \\ \langle\{11\}01\|R\|\{11\}01\rangle &= 0 \end{aligned} \tag{4.7}$$

$$\begin{aligned} \langle\{2\}11\|R\|\{2\}11\rangle &= 1 \\ \langle\{2\}11\|R\|\{2\}00\rangle &= \tfrac{1}{2} \\ \langle\{2\}00\|R\|\{2\}11\rangle &= \tfrac{3}{2} \\ \langle\{2\}00\|R\|\{2\}00\rangle &= 0\,. \end{aligned} \tag{4.8}$$

Our next problem is to write the matrix (3.25) explicitly with the columns characterized by $s\sigma t\tau$ and the rows by $s'\sigma' t'\tau'$. In principle we would have to write nine matrices as in $R_{qq'}, q = +, 0, -; q' = +, 0, -$. We note, however, that from the Clebsch–Gordan coefficients in (3.25) we have

$$q = \sigma' - \sigma \qquad q' = \tau' - \tau \tag{4.9}$$

and thus our first objective will be to identify what element $R_{qq'}$ appears in each square of our matrix from the selection rules (4.9).

(b) The energy spectrum for n = 2, {h} = {11}

We shall start with the case when $\{h\} = \{11\}$ and to avoid using the full notation $s\sigma t\tau$ for the column where $s = 1, t = 0$ (or $s = 0, t = 1$) we shall indicate σ, τ as $\sigma = 1, 0, -1, \tau = \bar{0}$ (or $\sigma = \bar{0}, \tau = 1, 0, -1$) , i.e. we put a bar on the 0 when the value of either the ordinary or sign spin is 0. The same convention will be followed for the rows $s'\sigma' t'\tau'$, and we shall order the values in columns and rows in a way that will be convenient for our later analysis.

We note that the first (second) number appearing in the upper row or left-hand column indicates always the projection of the ordinary (sign) spin. Thus we have the matrix

$\sigma\tau$ / $\sigma'\tau'$	$\bar{0}1$	$\bar{0}0$	$0\bar{0}$	$1\bar{0}$	$-1\bar{0}$	$\bar{0}-1$
$\bar{0}1$	0	R_{0+}	R_{0+}	R_{-+}	R_{++}	0
$\bar{0}0$	R_{0-}	0	0	0	0	R_{0+}
$0\bar{0}$	R_{0-}	0	0	0	0	R_{0+}
$1\bar{0}$	R_{+-}	0	0	0	0	R_{++}
$-1\bar{0}$	R_{--}	0	0	0	0	R_{-+}
$\bar{0}-1$	0	R_{0-}	R_{0-}	R_{--}	R_{+-}	0

(4.10)

where in each box we put only the $R_{qq'}$ that appear from relations (4.9) and the zeros are due to the fact that $R_{00}, R_{\pm 0}$, are not present in (3.26). The values of $R_{qq'}$ in each box with $q, q' = +, 0, -$ can be evaluated from (4.7) and the elementary Clebsch–Gordan coefficients in (3.25) given, e.g., in [23, page 225].

We are not only interested in the matrix elements of $R_{qq'}$ appearing in each of the blocks of (4.10), but also in the factor $\lambda_{qq'}$ of (3.27) accompanying them in the H of (3.26). Furthermore we wish to include also the $2T_3$ of (3.26) and deal with $H - nEI$, from which we will get the matrix operator that eventually leads to the secular equation determining the value of the energy. Thus our matrix (4.10) now takes the form

$\mathbf{M} \equiv$

$\sigma\tau$ / $\sigma'\tau'$	$\bar{0}1$	$\bar{0}0$	$0\bar{0}$	$1\bar{0}$	$-1\bar{0}$	$\bar{0}-1$
$\bar{0}1$	$2-2E$	0	$-\sqrt{2}k$	$2\mathrm{i}\omega\eta_+$	$2\mathrm{i}\omega\xi^+$	0
$\bar{0}0$	0	$-2E$	0	0	0	0
$0\bar{0}$	$-\sqrt{2}k$	0	$-2E$	0	0	$\sqrt{2}k$
$1\bar{0}$	$-2\mathrm{i}\omega\xi^+$	0	0	$-2E$	0	$2\mathrm{i}\omega\xi^+$
$-1\bar{0}$	$-2\mathrm{i}\omega\eta^+$	0	0	0	$-2E$	$2\mathrm{i}\omega\eta_+$
$\bar{0}-1$	0	0	$\sqrt{2}k$	$-2\mathrm{i}\omega\eta_+$	$-2\mathrm{i}\omega\xi^+$	$-2-2E$

(4.11)

where we already evaluated the matrix elements of $R_{qq'}\lambda_{qq'}$ with respect to the spin part of the state (3.21) using (3.25) and (4.7).

We now have to transform the operator matrix (4.11) into a numerical one. If it had been a 1×1 matrix we would have only to take the expectation value with the coresponding orbital part of the single particle state. If it is a full matrix, we have to consider the orbital part of the set of states (3.21) as forming a diagonal matrix whose rows and columns are enumerated in the same way as (4.11). As the latter contains only η_+, ξ^+ operators we can disregard in (3.21) the term η_-^ν, i.e. for simplicity take $\nu = 0$. Furthermore, $\exp(\mathrm{i}kx_3)$ in (3.21) is irrelevant, as equation (4.11) already contains the eigenvalue k. The orbital part of the ket (3.21) reduces then to the form

$$|\mu - \sigma\rangle = [(\mu - \sigma)!]^{-1/2}\eta_+^{\mu-\sigma}|0\rangle\,. \tag{4.12}$$

Using the compact notation introduced before

$$|\{11\}1\sigma, 00\rangle = |\sigma\bar{0}\rangle \qquad |\{11\}00, 1\tau\rangle = |\bar{0}\tau\rangle \tag{4.13}$$

we can enumerate rows and columns in the same way as in (4.11) and our diagonal matrix becomes

$$\mathbf{\Delta} = \begin{array}{|c|cccccc|}\hline \sigma\tau & \bar{0}1 & \bar{0}0 & 0\bar{0} & 1\bar{0} & -1\bar{0} & \bar{0}-1 \\ \sigma'\tau' & & & & & & \\ \hline \bar{0}1 & |\mu\rangle & & & & & \\ \bar{0}0 & & |\mu\rangle & & & & \\ 0\bar{0} & & & |\mu\rangle & & & \\ 1\bar{0} & & & & |\mu-1\rangle & & \\ -1\bar{0} & & & & & |\mu+1\rangle & \\ \bar{0}-1 & & & & & & |\mu\rangle \\ \hline \end{array}\,. \tag{4.14}$$

To transform the matrix (4.11) to numerical form we just have to replace $\mathbf{M}$ by $\mathbf{\Delta}^\dagger\mathbf{M}\mathbf{\Delta}$ carrying out the operations on the state (4.12), and thus get the matrix

$\mathbf{\Delta}^\dagger\mathbf{M}\mathbf{\Delta}$

$$\equiv \begin{array}{|c|cccccc|}\hline \sigma\tau & \bar{0}1 & \bar{0}0 & 0\bar{0} & 1\bar{0} & -1\bar{0} & \bar{0}-1 \\ \sigma'\tau' & & & & & & \\ \hline \bar{0}1 & 2-2E & 0 & -\sqrt{2}k & 2\mathrm{i}\omega\sqrt{\mu} & 2\mathrm{i}\omega\sqrt{\mu+1} & 0 \\ \bar{0}0 & 0 & -2E & 0 & 0 & 0 & 0 \\ 0\bar{0} & -\sqrt{2}k & 0 & -2E & 0 & 0 & \sqrt{2}k \\ 1\bar{0} & -2\mathrm{i}\omega\sqrt{\mu} & 0 & 0 & -2E & 0 & 2\mathrm{i}\omega\sqrt{\mu} \\ -1\bar{0} & -2\mathrm{i}\omega\sqrt{\mu+1} & 0 & 0 & 0 & -2E & 2\mathrm{i}\omega\sqrt{\mu+1} \\ \bar{0}-1 & 0 & 0 & \sqrt{2}k & -2\mathrm{i}\omega\sqrt{\mu} & -2\mathrm{i}\omega\sqrt{\mu+1} & 2-2E \\ \hline \end{array}\,. \tag{4.15}$$

Setting the determinant of the matrix (4.15) to 0 we get a secular equation that gives the eigenvalues of E as function of μ, k and ω.

Before proceeding to analyse the secular equation, it is convenient to note that if E is taken out of the matrix (4.15) the second row and column are zero, which implies that one of the values of the energy is $E = 0$ and the remaining matrix we have to analyse is of dimension 5×5, obtained from (4.15) when we suppress the second row and column. This is a dimensionality that we would have expected from the Duffin–Kemmer [5] analysis for the case of spin 0 if we replaced $\boldsymbol{p}$ by $\boldsymbol{p} - (e/2)(\boldsymbol{r} \times \boldsymbol{\mathcal{H}})$, as their matrices are also of dimension 5×5.

Returning to the matrix (4.15), an elementary calculation of its determinant shows that it gives the secular equation

$$E^4[4E^2 - 4 - 4k^2 - 8\omega^2(2\mu + 1)] = 0. \tag{4.16}$$

The appearance of the value $E = 0$ reflects the cockroach nest effect discussed in [24]. The relevant physical energy is given by the square bracket expression in (4.16) when equated to 0, leading to the equation

$$E^2 = 1 + k^2 + 2\omega^2(2\mu + 1). \tag{4.17}$$

We now compare (4.17) for $n = 2$, $\{h\} = \{11\}$ with (2.17) where $n = 1$. Note first that from (3.27) $e\mathcal{H}$ can be replaced by $2\omega^2$ in (2.17). Furthermore μ and n_+ are equivalent as both take integer values related with the number of quanta associated with the operator η_+. They differ though by the fact that in (2.17) appears the projection σ of the ordinary spin of value $\frac{1}{2}$ while it is absent in (4.17), which suggests that for $n = 2$, $\{h\} = \{11\}$ we are dealing with a particle of spin 0, instead of the spin $\frac{1}{2}$ we had for $n = 1$. This conclusion could also be expected from the 5×5 representation of Kemmer in a magnetic field.

Going to the non-relativistic limit whose energy we denote by ϵ we can write

$$E^2 - 1 = (E+1)(E-1) \simeq 2\epsilon \tag{4.18}$$

and thus from (4.17) we have

$$\epsilon = \tfrac{1}{2}k^2 + \omega^2(2\mu + 1) \tag{4.19}$$

which is precisely the result for a particle without spin [25].

(c) The energy spectrum for n = 2, {h} = {2}

As we mentioned in subsection (a), in the case that $\{h\} = \{2\}$ the states (4.2) could either have $s = 1, t = 1$ or $s = 0, t = 0$ and in (4.8) we gave the reduced matrix elements of R corresponding to this case.

The projections of the ordinary and sign spins are

$$\begin{aligned} &s = 1 \quad \sigma = 1, 0, -1 \qquad t = 1 \quad \tau = 1, 0, -1 \\ &s = 0 \quad \sigma = 0 \qquad t = 0 \quad \tau = 0\,. \end{aligned} \tag{4.20}$$

As in the previous section we shall designate the projection 0 of $s = t = 0$ by a bar above and thus $\sigma\tau$ for $s = t = 0$ will be denoted as $\bar{0}\bar{0}$, while when $s = t = 1$ we simply put the corresponding numbers for $\sigma\tau$.

Altogether we would have then 10 possibilities for the pairs $\sigma\tau$ and for convenience in our future analysis we shall order them as indicated below:

$$01, 10, -10, 0-1; 11, -1-1, 00, \bar{0}\bar{0}, 1-1, -1-1. \tag{4.21}$$

Our first objective in the calculation would be to put these pairs of numbers in the order indicated in (4.21) in the row and column of a matrix of the type (4.10). This would allow us to put in each box a

$$R_{qq'} = R_{\sigma'-\sigma,\tau'-t} \tag{4.22}$$

as was done in (4.10) and where we made use of (4.9). As the process is trivial we do not write explicitly the 10×10 matrix corresponding to the 6×6 one in (4.10).

Once we have though this matrix we have to multiply $R_{qq'}$ by $\lambda_{qq'}$ and obtain a matrix $\mathbf{M}$ equivalent to (4.11). Finally we have to write the orbital part of the states of the problem in the diagonal matrix form $\mathbf{\Delta}$ of (4.14) but now with 10 components, and consider the product equivalent to $\mathbf{\Delta}^\dagger\mathbf{M}\mathbf{\Delta}$ of (4.15) to obtain the final 10×10 numerical matrix.

As we are mainly interested in the behaviour of the energy as function the quantum number μ and the frequency ω, we shall only discuss the case when $k = 0$, i.e. when the particle is at rest in the direction of the magnetic field. In that case we see from equation (3.23) that only terms of the form $R_{\pm\pm}$ remain as $R_{0\pm}$ have coefficients 0. This implies another integral of motion as from (4.22) we have either

$$\sigma + \tau \qquad \sigma' + \tau' \qquad \text{both even} \tag{4.23}$$

or

$$\sigma + \tau \qquad \sigma' + \tau' \qquad \text{both odd}\,. \tag{4.24}$$

Thus our 10×10 matrix breaks in two blocks along the diagonal, one in which $\sigma + \tau, \sigma' + \tau'$ in column and row are odd, i.e. when they take the first four values in

(4.21), and another when $\sigma + \tau, \sigma' + \tau'$ are even when they take the last six values of (4.21). We proceed then to give these two numerical matrices explicitly:

$\sigma+\tau$ odd: (4.25)

$\sigma'\tau'$ \ $\sigma\tau$	01	10	-10	$0-1$
01	$2-2E$	$2\mathrm{i}\omega\sqrt{\mu}$	$-2\mathrm{i}\omega\sqrt{\mu+1}$	0
10	$-2\mathrm{i}\omega\sqrt{\mu}$	$-2E$	0	$-2\mathrm{i}\omega\sqrt{\mu}$
-10	$2\mathrm{i}\omega\sqrt{\mu+1}$	0	$-2E$	$2\mathrm{i}\omega\sqrt{\mu+1}$
$0-1$	0	$2\mathrm{i}\omega\sqrt{\mu}$	$-2\mathrm{i}\omega\sqrt{\mu+1}$	$-2-2E$

$\sigma+\tau$ even: (4.26)

$\sigma'\tau'$ \ $\sigma\tau$	11	-11	00	$\bar{0}\bar{0}$	$1-1$	$-1-1$
11	$2-2E$	0	$-2\mathrm{i}\omega\sqrt{\mu}$	$-2\mathrm{i}\omega\sqrt{\mu}$	0	0
-11	0	$2-2E$	$2\mathrm{i}\omega\sqrt{\mu+1}$	$-2\mathrm{i}\omega\sqrt{\mu+1}$	0	0
00	$2\mathrm{i}\omega\sqrt{\mu}$	$-2\mathrm{i}\omega\sqrt{\mu+1}$	$-2E$	0	$2\mathrm{i}\omega\sqrt{\mu}$	$-2\mathrm{i}\omega\sqrt{\mu+1}$
$\bar{0}\bar{0}$	$2\mathrm{i}\omega\sqrt{\mu}$	$2\mathrm{i}\omega\sqrt{\mu+1}$	0	$-2E$	$-2\mathrm{i}\omega\sqrt{\mu}$	$-2\mathrm{i}\omega\sqrt{\mu+1}$
$1-1$	0	0	$-2\mathrm{i}\omega\sqrt{\mu}$	$2\mathrm{i}\omega\sqrt{\mu}$	$-2-2E$	0
$-1-1$	0	0	$2\mathrm{i}\omega\sqrt{\mu+1}$	$2\mathrm{i}\omega\sqrt{\mu+1}$	0	$-2-2E$

Setting the determinants of the above matrices to 0, we get the secular equations from which the energy can be obtained as function of μ and ω, and also some information on the spin with which the energies are associated.

In the case (4.25) when $\sigma + \tau$ is odd the secular equation is

$$E^2\{E^2 - 1 - 2\omega^2(2\mu+1)\} = 0 \tag{4.27}$$

which has exactly the same form for partition {2} than for {11} in (4.17). When $\omega, \mu \ll 1$ we can obtain the non-relativistic limit, and looking at (4.25) we see that main part of the state is in the first block whose projection of the ordinary spin is 0, but the spin itself is, from (4.2), $s = 1$. So it is the zero projection of the spin, in the direction of the magnetic field, that is responsible for the similarity between (4.17) and (4.27), and not the spin itself.

In the case (4.26) when $\sigma + \tau$ is even the matrix is 6×6 and the secular equation in more complex, but it can be obtained straightforwardly giving rise to the expression

$$E^6 - 2E^4[1 + 2\omega^2(2\mu+1)] + E^2[1 + 2\omega^2(2\mu+1)]^2 - 4\omega^4 = 0 \tag{4.28}$$

which is a cubic equation in E^2, but in which the term $[1 + 2\omega^2(2\mu+1)]$ also appears.

Again in the non-relativistic limit we have a behaviour similar to (4.19) but now from the matrix (4.26) we see the state is associated with $\sigma = 1$, so the spin besides being 1 also has a projection 1 and this complicates greatly the spectrum as function of μ, ω as is clear from (4.28).

We have seen that the results for the energy and the spin coincide in the approach we have followed, with those of Kemmer [5] and Duffin when in their formulation of relativistic particles of spin 0 and 1, one introduces a magnetic field. It is interesting to note though that our procedure is applicable to any value n, and thus to particles that can go up to spin $\frac{1}{2}n$, and to stress this generality we shall consider in the next subsection the case $n = 3$.

(d) The energy spectrum for n = 3, {h} = {3}

From the discussion in section 3, the only part of our analysis that requires care for an arbitrary n, is the matrix element (3.25) which, in the particular problem indicated in the title of this subsection, is given by

$$\langle\{3\}s'\sigma't'\tau'|R_{qq'}|\{3\}s\sigma t\tau\rangle. \tag{4.29}$$

Because we are dealing with a symmetric representation of the U(4) group there is no need of the extra indices γ', γ and besides $s' = t', s = t$ and they can only have the values $\frac{3}{2}$ and $\frac{1}{2}$.

Again because of the symmetric nature of the representation {3} of U(4) or, equivalently, of the group S(3), the expression (4.29) can be written as

$$3\langle\{3\}s'\sigma't'\tau'|s_q(1) \otimes t_{q'}(1)|\{3\}s\sigma t\tau\rangle. \tag{4.30}$$

To evaluate (4.30) it is convenient to decompose bra and ket in the part that have index 1 in ordinary and sign spin and the rest that has to do with indices 2 and 3. This can be achieved with the help of reduced Wigner coefficient of U(4) or, equivalently, of an appropriate fractional parentage coefficient as discussed, among other papers, in [22]. Once this decomposition is achieved the matrix element (4.30) can be derived using standard Racah analysis.

After this step for the case $n = 3$, $\{h\} = \{3\}$ we just follow the same procedure that we discussed previously for the case $\{h\} = \{11\}$ and $\{h\} = \{2\}$. We use a similar notation for the projection of the ordinary and sign spin as we did before, i.e.

$$\begin{array}{lll} s = t = \frac{3}{2} & \sigma = \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2} & \tau = \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2} \\ s = t = \frac{1}{2} & \sigma = \overline{\tfrac{1}{2}}, \overline{-\tfrac{1}{2}} & \tau = \overline{\tfrac{1}{2}}, \overline{-\tfrac{1}{2}}\,. \end{array} \tag{4.31}$$

In principle we have 20 possibilities for the values of $\sigma\tau$ from (4.31), but again we shall restrict ourselves to the case $k = 0$, i.e. when the particle is at rest in the direction of the magnetic field, and then there is the selection rule that $\sigma + \tau$ is either odd or even and this reduces the matrices to 10×10. We give in (4.32) a table of the numerical matrix when $\sigma + \tau$ is odd, from which we could extract the secular equation by setting its determinant to 0. We shall not carry out this last step as it clearly gives an equation, of the 10th degree in E whose analysis is not simple. Rather we wish only to stress that we have a technique that allows us to obtain and solve the relativistic equation for a particle with arbitrary spin subjected to an interaction by reducing it to a supermultiplet type of problem where we have the ordinary and sign spins and thus a U(4) type of symmetry.

5. Conclusions

The example discussed in this paper was one of a free particle with arbitrary spin in a magnetic field because, when formulated in its matrix representation, it always gives rise to finite matrices and thus its solution, at least with the use of computers, will give us exact values for the energies.

The formalism though can be extended to a particle with arbitrary spin in any potential and, in particular, to the Coulomb one. The analysis has then to be carried in a variational manner, and in its more convenient form, through the use of harmonic oscillator states for the orbital part of the problem, while the ordinary- and sign-spin part continue to be given by basis of irreducible representations of the U(4) group.

The final results for the energy levels would of course not be exact, but going to a sufficient number of quanta they could give reasonable approximations to the way in which the spin of the particle affects the spectra of the problem.

This program is being implemented at the present time together with other aspects involving the representation of the Lorentz group in the problem, and this paper is thus part I of a series.

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Appendix: Commutation relations of the generators of U(4)

The generators of u(4) are given in (3.9) to which we have to add the unit operator $\hat{I} \otimes \check{I}$. As the operators s_{iu}, t_{iu} commute with s_{iv}, t_{iv} if $v \neq u$, it would be enough to prove the commutation relation (3.11) by considering the corresponding ones for the operators

$$s_i \otimes \check{I} \qquad \hat{I} \otimes t_i \qquad s_i \otimes t_j. \tag{A.1}$$

As we have the following relations:

$$\begin{aligned} &[s_i, s_j] = \mathrm{i}\epsilon_{ijk} s_k \qquad [t_i, t_j] = \mathrm{i}\epsilon_{ijk} t_k \\ &s_i s_j = \tfrac{1}{4}\delta_{ij} + \tfrac{1}{2}\mathrm{i}\epsilon_{ijk} s_k \qquad t_i t_j = \tfrac{1}{4}\delta_{ij} + \tfrac{1}{2}\epsilon_{ijk} t_k \end{aligned} \tag{A.2}$$

we immediately conclude that

$$\begin{aligned} &[s_i \otimes \check{I}, s_j \otimes \check{I}] = \mathrm{i}\epsilon_{ijk} s_k \otimes \check{I} \qquad [\hat{I} \otimes t_i, \check{I} \otimes t_j] = \mathrm{i}\epsilon_{ijk} \hat{I} \otimes t_k \\ &[s_i \otimes \check{I}, \hat{I} \otimes t_j] = 0 \qquad [s_i \otimes \check{I}, s_j \otimes t_k] = \mathrm{i}\epsilon_{ij\ell} s_\ell \otimes t_k \\ &[\hat{I} \otimes t_i, s_j \otimes t_k] = \mathrm{i}\epsilon_{ik\ell} s_j \otimes t_\ell \\ &[s_i \otimes t_j, s_k \otimes t_\ell] = s_i s_k \otimes t_j t_\ell - s_k s_i \otimes t_\ell t_j \\ &\qquad = \left[\tfrac{1}{4}\delta_{ik} + \tfrac{1}{2}\mathrm{i}\epsilon_{ikm} s_m\right] \otimes \left[\tfrac{1}{2}\delta_{j\ell} + \tfrac{1}{2}\mathrm{i}\epsilon_{j\ell n} t_n\right] \\ &\qquad\quad - \left[\tfrac{1}{4}\delta_{ki} + \tfrac{1}{2}\mathrm{i}\epsilon_{kim} s_m\right] \otimes \left[\tfrac{1}{4}\delta_{\ell j} + \tfrac{1}{2}\mathrm{i}\epsilon_{\ell jn} t_n\right] \\ &\qquad = \tfrac{1}{4}\mathrm{i}\epsilon_{ikm} s_m \delta_{j\ell} + \tfrac{1}{4}\delta_{ik}\mathrm{i}\epsilon_{j\ell n} t_n \end{aligned} \tag{A.3}$$

and from (A.3) the commutation relations (3.11) follow immediately.

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